Distribution Theory.

Recall. If X is top. space, then its dual X^* is the space of continuous linear functionals $X \to \mathbb{C}$.

Rem. If $Y \subseteq X$, then $X^* \subseteq Y^*$.

Ex1. For $1 \leq p < \infty$, $X = L^p(\mathbb{R}^n) \Rightarrow$ $X^* = L^q(\mathbb{R}^n)$ where $q = p^*$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

Ex2. If $X = C_0(\mathbb{R}^n)$ then $X^* = \mathcal{M}(\mathbb{R}^n)$, cplx measures on $\mathbb{R}^n$. Note that in this case, X^* is not representable as space of fcns on $\mathbb{R}^n$; $\mu \in \mathcal{M} \Rightarrow \mu : \mathcal{B}_{\mathbb{R}^n} \to \mathbb{C}$.

The spaces of distributions $\mathcal{D}'$, $\mathcal{S}'$ are duals of $\mathcal{D} := \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\mathcal{S}$, respectively, with suitable topologies.

<u>The case of $\mathcal{D} = \mathcal{C}_c^\infty = \mathcal{C}_c^\infty(\mathbb{R}^n)$.</u>

For every $K \subset\subset \mathbb{R}^n$, $\mathcal{C}_c^\infty(K)$ is the Fréchet space of $\mathcal{C}^\infty$ fcns φ on $\mathbb{R}^n$ w/ $\operatorname{supp}\varphi \subseteq K$ and s.t. the top. is generated by the family of semi-norms

$$\|\varphi\|_{\mathcal{C}^k(K)} = \sum_{|\alpha| \le k} \sup_K |\partial^\alpha \varphi(x)|, \quad k \in \mathbb{Z}_+.$$

$$\Rightarrow \mathcal{C}_c^\infty(\mathbb{R}^n) = \bigcup_{K \subset\subset \mathbb{R}^n} \mathcal{C}_c^\infty(K).$$

There is a LC top. on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ s.t.

(i) $\varphi_j \to \varphi$ in $\mathcal{C}_c^\infty(\mathbb{R}^n) \Leftrightarrow \varphi_j, \varphi \in \mathcal{C}_c^\infty(K)$ for some $K \subset\subset \mathbb{R}^n$ and $\varphi_j \to \varphi$ in $\mathcal{C}_c^\infty(K)$

(ii) A linear map $T: \mathcal{C}_c^\infty \to X$, where X is LC vector space, is cont. $\Leftrightarrow$ $T\varphi_j \to T\varphi$ whenever $\varphi_j \to \varphi$ in $\mathcal{C}_c^\infty(K)$ in some $K \subset\subset \mathbb{R}^n$.

<u>Def</u> The <u>space of distributions</u>, $\mathcal{D}'$, is the space of cont. linear functionals $F: \mathcal{C}_c^\infty \to \mathbb{C}$ endowed w/ weak* topology, i.e. $F_j \to F$ in $\mathcal{D}' \Leftrightarrow F_j\varphi \to F\varphi \quad \forall \varphi \in \mathcal{C}_c^\infty$.

Ex. ① $f \in L^p(\mathbb{R}^n) \Rightarrow$

$F\varphi = \int_{\mathbb{R}^n} f\varphi\, dx$ defines $F \in \mathcal{D}'$.

Check. Suppose $\varphi_j \to \varphi$ in $C_c^\infty \Leftrightarrow$ $\varphi_j \to \varphi$ in $C_c^\infty(K)$ for some $K \subset\subset \mathbb{R}^n$

$$|F\varphi - F\varphi_j| \le \int_K |f|\,|\varphi - \varphi_j|\, dx$$

$$\le \|f\|_{L^p} \|\varphi - \varphi_j\|_{L^q(K)} \to 0$$

$\frac{1}{p} + \frac{1}{q} = 1$

② $\mu \in \mathcal{M}(\mathbb{R}^n) \Rightarrow F\varphi = \int \varphi\, d\mu$ defines $F \in \mathcal{D}'$.

Check. $|F\varphi_j - F\varphi| \le \int |\varphi - \varphi_j|\, d\mu$

$$\le \|\varphi - \varphi_j\|_u\, \mu(K)$$

$$\to 0.$$

(3) Let $n=1$ and $k\in\mathbb{Z}_+$. Then
$$F\varphi = \varphi^{(k)}(0) = \frac{d^k\varphi}{dx^k}(0), \quad \varphi\in C_c^\infty$$
defines $F\in\mathcal{D}'$.

<u>Check.</u> $\varphi_j\to\varphi$ in C_c^∞ $\Rightarrow$
$$F\varphi_j = \varphi_j^{(k)}(0) \to \varphi^{(k)}(0) = F\varphi$$

(4) Let $P(x,\partial) = \sum_{|\alpha|\le m} a_\alpha(x)\,\partial^\alpha$ be a partial diff. oper. (PDO). Then
$$P\varphi = P(x,\partial)\varphi, \quad \varphi\in C_c^\infty$$
defines a <u>cont. linear map</u> $C_c^\infty \circlearrowleft$.

<u>Check</u>: $\varphi_j\to\varphi$ in C_c^∞. Then $\exists K\subset\subset\mathbb{R}^n$ s.t. $\operatorname{supp}\varphi_j, \operatorname{supp}\varphi \subseteq K$ and $\partial^\alpha\varphi_j\to\partial^\alpha\varphi$. $\Rightarrow$ $\operatorname{supp}P\varphi_j, \operatorname{supp}P\varphi\subseteq K$ and $\partial^\beta P\varphi_j \to \partial^\beta P\varphi$, $\forall\beta$.

<u>Def.</u> For $F \in \mathcal{D}'$, the <u>support</u> of F, $\operatorname{supp} F \subseteq \mathbb{R}^n$, is the closed set s.t. $U \underset{\text{open}}{\subseteq} (\operatorname{supp} F)^c$ if

$F\varphi = 0$, $\forall \varphi \in C_c^\infty(U)$.

<u>Prop 1.</u> If $V \subseteq (\operatorname{supp} T)^c$, then $F\varphi = 0$, $\forall \varphi \in C_c^\infty(V)$.

Pf. $V = \bigcup_\alpha U_\alpha$, $F\varphi = 0$ $\forall \varphi \in C_c^\infty(U_\alpha)$ by def. of supp F. Pick $\varphi \in C_c^\infty(V)$ w/ $K = \operatorname{supp}\varphi \subset\subset V$. Then $K \subseteq \bigcup_{j=1}^m U_{\alpha_j}$

By a partition of unity, $\exists \psi_j \in C_c^\infty(U_{\alpha_j})$ s.t. $\sum_{j=1}^m \psi_j = 1$ on K.

Then, $\varphi = \sum_1^m \varphi\psi_j$, $F\varphi = \sum_1^m F\varphi\psi_j = 0$ since $\varphi\psi_j \in C_j^\infty(U_{\alpha_j})$. ∎

Notational convention. One often uses the notation $\langle F, \varphi \rangle = F\varphi$ to reflect that if $F = F_\psi$ is given by $\psi \in \mathcal{C}_c^\infty$, then $\langle F_\psi, \varphi \rangle = \int \psi\varphi \, dx = \langle F_\varphi, \psi \rangle$. Often simply $\langle \psi, \varphi \rangle = \langle \varphi, \psi \rangle$.

General Procedure of extending cont. linear maps $T : \mathcal{C}_c^\infty \to \mathcal{C}_c^\infty$ to $\tilde{T} : \mathcal{D}' \to \mathcal{D}'$.

First, suppose for $\varphi, \psi \in \mathcal{C}_c^\infty$

$$\langle T\psi, \varphi \rangle = \int (T\psi)\varphi = \int \psi T'\varphi = \langle \psi, T'\varphi \rangle$$

Then, we define $\tilde{T}F$ by

$$\langle \tilde{T}F, \varphi \rangle = \langle F, T'\varphi \rangle.$$

Then $\tilde{T}\psi = T\psi$ for $\psi \in \mathcal{C}_c^\infty$

since $\langle \tilde{T}\psi, \varphi\rangle = \langle \psi, T'\varphi\rangle =$

$= \langle T\psi, \varphi\rangle, \quad \forall \varphi \in C_c^\infty.$

Thus, $\tilde{T}\psi = T\psi$ in $C_c^\infty \subseteq \mathcal{D}'$.

Ex. (1) For $\alpha \in C_c^\infty$, $T\varphi = \alpha\varphi$.

Then, $\int (T\psi)\varphi = \int (\alpha\psi)\varphi = \int \psi(\alpha\varphi)$

so $T' = T$ and αF is defined

for $F \in \mathcal{D}'$ by $\langle \alpha F, \varphi\rangle = \langle F, \alpha\varphi\rangle$.

(2) $T\varphi = \alpha * \varphi = \varphi * \alpha$. Then

$\langle T\psi, \varphi\rangle = \int (\psi * \alpha)\varphi =$

$\int \left(\int \psi(y)\alpha(x-y)\,dy \right) \varphi(x)\,dx =$

$\int \left(\int \alpha(x-y)\varphi(x)\,dx \right) \psi(y)\,dy =$

$= \int \psi(y)(\tilde{\alpha} * \varphi)(y)\,dy$, where $\tilde{\alpha}(y) = \alpha(-y)$. Thus, we define $F * \alpha \in \mathcal{D}'$ by $\langle F * \alpha, \varphi \rangle = \langle F, \tilde{\alpha} * \varphi \rangle$.

<u>Rem</u> Note $\int (\tau_y \psi)\varphi\,dx = \int \psi(x-y)\varphi(x)\,dx$
$= \int \psi(x')\varphi(x'+y)\,dx' = \int \psi(\tau_{-y}\varphi)\,dx$

Thus, one could define $F * \alpha$ as $\langle F, \tau_y \tilde{\alpha} \rangle$, which gives a function $\mathbb{R}^n \to \mathbb{C}$.

③ For $\varphi, \psi \in C_c^\infty$, integration by parts $\Rightarrow \int \varphi(\partial_j \psi) = -\int (\partial_j \varphi)\psi \Rightarrow$
$\int \varphi(\partial^\alpha \psi) = (-1)^{|\alpha|} \int (\partial^\alpha \varphi)\psi$. Thus, we define $\partial^\alpha F$ by using $T' = (-1)^{|\alpha|}\partial^\alpha$:
$\langle \partial^\alpha F, \varphi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \varphi \rangle$.

Prop 2. For $F \in \mathcal{D}'$, $\psi \in C_c^\infty$, consider $f(x) = \langle F, \tau_x \tilde{\psi} \rangle$. Then

(i) $f \in C^\infty$ and $\partial^\alpha f = \langle \partial^\alpha F, \tau_x \tilde{\psi} \rangle$.

(ii) $\underbrace{\langle f, \varphi \rangle = \langle F * \psi, \varphi \rangle}^{\forall \varphi \in C_c^\infty} \left(= \langle F, \varphi * \tilde{\psi} \rangle \right)$.

Rem. As a consequence, we can define $F * \psi$ either by $\langle F * \psi, \varphi \rangle = \langle F, \varphi * \tilde{\psi} \rangle$ or by $(F * \psi)(x) = \langle F, \tau_x \tilde{\psi} \rangle$. As a cons. of (i), have $\partial^\alpha (F * \psi) = \partial^\alpha F * \psi = F * \partial^\alpha \psi$.

Pf of Prop 2. We shall first prove

Lemma 1. Let $g = g(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $F \in \mathcal{D}'(\mathbb{R}^n)$.

(a) Suppose$^{(*)}$ $\forall X \subset\subset \mathbb{R}^n$ $\exists K_X \subset\subset \mathbb{R}^n$ s.t. $\operatorname{supp} g(x, \cdot) \subseteq K_X$, $\forall x \in X$. Then, $h(x) = \langle F, g(x, \cdot) \rangle \in C^\infty$, $\partial^\alpha h(x) = \langle F, \partial_x^\alpha g(x, \cdot) \rangle$.

(b) Suppose $g \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then $h \in C_c^\infty$ and

$$\langle F, h \rangle \overset{?}{=} \langle F, \int g(\cdot, y)\, dy \rangle = \int \langle F, g(\cdot, y) \rangle\, dy.$$

Pf. (a) We shall do $n=1$. Suffices to show that h is differentiable at $x_0 \in \mathbb{R}$.

$$\frac{h(x_0+\delta)-h(x_0)}{\delta} = \langle F, \frac{g(x_0+\delta, \cdot) - g(x_0, \cdot)}{\delta} \rangle.$$

(*) It is straightforward to verify that

$$\frac{g(x_0+\delta, \cdot) - g(x_0, \cdot)}{\delta} \xrightarrow[\delta \to 0]{} g_x(x_0, \cdot) \text{ in } C_c^\infty(\mathbb{R}^n) \text{ for each } x \in \mathbb{R}^n.$$

$\Rightarrow$ h is differentiable at x_0 and

$$h'(x_0) = \langle F, g_x(x_0, \cdot) \rangle.$$

Repeating this shows $h \in C^k$, for any k, i.e., $h \in C^\infty$, and $h^{(k)}(x) = \langle F, \partial_x^k g(x_0, \cdot) \rangle$ as desired. (General n is basically the same argument.)

(b) Let $\operatorname{supp} g \in Q_x \times Q_y$, where Q_x, Q_y are cpt cubes $\mathbb{R}^n$. Let $\varphi(x) = \int g(x,y)\,dy$. By Thm 2.27, have $\varphi \in C^\infty$, $\partial^\alpha \varphi = \int \partial_x^\alpha g(x,y)\,dy$,

and $\operatorname{supp} \varphi \subset\subset Q_x$. Since g is unif. cont. on $Q_x \times Q_y$, we may approximate $\int g(x,y)\,dy$ by Riemann sums. For $m \in \mathbb{Z}_+$, we partition Q_y into 2^{nm} equilateral cubes w/ centers $\{y_\ell^m\}_{\ell=1}^{2^{nm}}$ and volume $V_m \to 0$ as $m \to \infty$. Then

$$S_m(x) = \sum_{\ell=1}^{2^{nm}} g(x, y_\ell^m)\, V_m \to \int g(x,y)\,dy$$

unif. (in x).

Moreover, $S_m \in C_c^\infty(Q_x)$, $\partial^\alpha S_m = \sum_\ell \partial_x^\alpha g(x, y_\ell^m) \to \int \partial_x^\alpha g(x,y)\,dy$ unif. (in x). I.e., $S_m \to \varphi = \int g(\cdot, y)\,dy$ in C_c^∞. $\Rightarrow$

$\Rightarrow$

$\langle F, S_m \rangle \to \langle F, \int g(\cdot,y)\,dy \rangle$ as $m \to \infty$

On the other hand,

$$\langle F, S_m \rangle = \sum_{l=1}^{2^{nm}} \langle F, g(\cdot, y_l^m) \rangle V_m .$$

By (a) and $g(\cdot,y)=0$ if $y \notin Q_y$, have

$\langle F, g(\cdot,y) \rangle \in C_c^\infty$, $\partial^\alpha \langle F, g(\cdot,y) \rangle =$

$\langle F, \partial_y^\alpha g(\cdot,y) \rangle$, and clearly

$\operatorname{supp} \langle F, g(\cdot,y) \rangle \subseteq Q_y$. Thus,

$$\langle F, S_m \rangle = \sum_l \langle F, g(\cdot, y_l^m) \rangle V_m \longrightarrow$$

$$\int \langle F, g(\cdot,y) \rangle \, dy .$$

This proves (b) and Lemma 1 is proved.

Now, Prop. 2 follows from Lemma 1.

<u>For (i)</u>. Let $g(x,y) = \tau_x \tilde{\psi}(y) = \psi(x-y)$. Then $g \in C^\infty$ and satisfies condition in Lemma 1 (a). Thus, Prop 2 (i) follows.

$\{y : x \in X, y-x \in \operatorname{supp}\psi\}$

<u>For (ii)</u>.

$\operatorname{supp} g \subseteq X \times K_X$ w/ $X = \operatorname{supp}\varphi$.

Pick $\varphi \in C_c^\infty$ and let $g(x,y) = \varphi(y)\psi(y-x)$.

$$\Rightarrow \langle F, \varphi * \tilde{\psi} \rangle = \langle F, \int \varphi(y)\psi(y-x)\, dy \rangle$$

$$= \int \varphi(y) \langle F, \psi(y - \cdot\,) \rangle\, dy$$

$$= \int \langle F, \tau_y \tilde{\psi} \rangle \varphi(y)\, dy.$$

∎

Approximation in $\mathcal{D}$.

Let $\psi \in C_c^\infty$ and $\psi_t(x) = \frac{1}{t^n}\psi(x/t)$ as before.

<u>Obs.</u> $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < t < \delta$
$\Rightarrow \operatorname{supp} \psi_t \subseteq B(0,\varepsilon) = \{x : |x| < \varepsilon\}$.

Thus, if $\varphi \in C_c^\infty$, $K = \operatorname{supp}\varphi$ and $K \subseteq U$ open $\exists \delta > 0$ s.t.
$\operatorname{supp} \varphi * \psi_t \subset\subset U \quad \forall\, 0 < t < \delta$.

If we additionally assume $\int \psi\, dx = 1$, then by previous we get
$\varphi * \psi_t \to \varphi$ in C_c^∞.

It follows that $F * \varphi_t \to F$ in $\mathcal{D}'$. Prop 2 $\Rightarrow$ $F * \varphi_t \in C^\infty$, and if supp F is compact, then supp$(F * \varphi_t)$ also compact.

Finally, we note that if $F \in \mathcal{D}'$ and $K_j \nearrow \mathbb{R}^n$ compacts, $\chi_j \in C_c^\infty$ s.t. $\chi_j = 1$ on K_j (C^∞-Urysohn), then $F_j = \chi_j F \to F$ in $\mathcal{D}'$.

To summarize:

<u>Thm 1</u>. C_c^∞ is dense in $\mathcal{D}'$. In fact, if $\int \varphi \, dx = 1$, then $\chi_j F * \varphi \in C_c^\infty$, $\chi_j F * \varphi_{t_j} \to F$ for $t_j \to 0$.